International Journal of Applied Mathematics
& Statistical Sciences (IJAMSS)
ISSN 2319-3972
Vol. 2, Issue 4, Sep 2013, 53-62


International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

# A MARKOVIAN SINGLE SERVER QUEUEING SYSTEM WITH ARRIVALS DISCOURAGED BY QUEUE LENGTH

**A. SRIDHAR[1] & R. ALLAH PITCHAI[2]**

[1]Department of Mathematics, Government Thirumagal Mills College, Gudiyatham, Tamil Nadu, India

[2]Department of Mathematics, C. Abdul Hakeem College, Melvisharm, Tamil Nadu, India

## ABSTRACT

We study the queueing system where every customer on arrival makes one of the possible decisions either to join the queue or to go away without taking service never to return. Assuming such a decision to be entirely governed by the queue size at the instant of customer arrival, the transient solution is obtained analytically using the iteration method for a state dependent birth-death queue in which potential customers.

**KEYWORDS:** State Dependent Queues, Birth- Death Models, Transient State

## 1. INTRODUCTION

In the study of queueing systems the emphasis has been on obtaining steady state solution as it is simple to derive and straightforward techniques can be employed. Time-dependent analysis helps us to understand the behaviour of a system when the parameters involved are perturbed and it can contribute to the costs and benefits of operating a system. In addition, such transient analysis is useful in obtaining optimal solutions which lead to the control of the system. There has been a resurgence of interest in the time dependent analysis of birth-death queueingmodels. The exact time dependent analysis of the state dependent queueing systems is usually difficult and often impossible. Even in the simple M/M/1 queue which is a birth and death process with constant birth and death rates, analytical solution involves an infinite series of Bessel functions and their integrals[3,5]. In the real world problems theunderlying birth and death rates are complex and the difficulty is compounded in the transient analysis of such models.

This discouraged arrivals single server queueing system is useful to model a computing facility that is solely dedicated to batch-job processing. Queues with discouraged arrivals have applications in computers with batch job processing where job submissions are discouraged when the system is used frequently and arrivals are modeled as a Poisson process with state dependent arrival rate. The discouragement affects the arrival rate of the queueing system. Mores [2] considers discouragement in which the arrival rate falls according to a negative exponential law. We consider a single server queueing system in which the customers arrive in a Poisson fashion with rate depending on the number of the customers present in the system at that time. That is $\lambda_n = \dfrac{\lambda}{n+1}$, n=0,1,2,… and the time taken to process each customer is exponentially distributed with a constant service rate regardless of the number of customers in the system. That is $\mu_n = \mu$, n=1,2,3,….Ammaretal.[1] study single server finite capacity Markovian queue with discouraged arrivals and reneging using matrix method. Wang et al.[6] study extensive review on queueing systems with impatient customers.

In this paper, the transient solution to a state dependent birth-death queueing model in which potential customers are discouraged by queue length is obtained using iteration method.

The formulation of the queueing model is given in section 2. The steady state probability and some measures of discouraged arrivals queues in both finite and infinite cases are obtained in section 3. In section 4 steady state probability of infinite capacity queue is obtained. Some numerical illustrations are provided in section 5. Finally section 6 concludes the paper.

## 2. FORMULATION OF THE QUEUEING MODEL

In this section, we formulate the queueing model. The Markovianqueueing model investigated in this paper based on the following assumptions:

- We consider a single server queueing system in which the customers arrive in a Poisson fashion with rate that depends on the number of customers present in the system at that time is $\frac{\lambda}{n+1}$.
- The service times are independently, identically and exponentially distributed with parameter $\mu$.
- The customers are served in order of their arrival.

## 3. DISCOURAGED ARRIVALS QUEUE

Let $P_n(t)$ ,n=0,1,2,… be the probability that there are n customers in the system at time t. Then, the differential-difference equations are derived by using the general birth and death arguments. These equations are solved iteratively in steady state in order to obtain the steady state solution.

$$\frac{d}{dt}\left(P_0(t)\right) = \mu P_1(t) - \lambda P_0(t)$$

$$\frac{d}{dt}\left(P_n(t)\right) = \mu P_{n+1}(t) - \left(\frac{\lambda}{n+1} + \mu\right)P_n(t) + \frac{\lambda}{n} P_{n-1}(t),\ n \geq 1 \tag{1}$$

### 3.1. FINITE CAPACITY QUEUE WITH DISCOURAGEMENT

If the waiting room is the finite capacity N, that is the queue has at the most N customers in it, then the equations (1) are to be replaced by the following system of differential difference equations of the model are

$$\frac{d}{dt}\left(P_0^{(N)}(t)\right) = \mu P_1^{(N)}(t) - \lambda P_0^{(N)}(t)$$

$$\frac{d}{dt}\left(P_n^{(N)}(t)\right) = \mu P_{n+1}^{(N)}(t) - \left(\frac{\lambda}{n+1} + \mu\right)P_n^{(N)}(t) + \frac{\lambda}{n} P_{n-1}^{(N)}(t)$$

for, n=1,2,…,N-1

$$\frac{d}{dt}\left(P_N^{(N)}(t)\right) = \frac{\lambda}{N} P_{N-1}^{(N)}(t) - \mu P_N^{(N)}(t) \tag{2}$$

Where the superscripts emphasize the fact that we are dealing with the case of finite capacity N of the waiting room.

In steady state, $\lim_{t\to\infty} P_n^{(N)}(t) = P_n$ and therefore , $\frac{dP_n^{(N)}(t)}{dt} = 0$ as $t \to \infty$ .

**Theorem 3.1**

If the steady state equations of the queueing model are

$$0 = \mu P_1^{(N)} - \lambda P_0^{(N)} \tag{3}$$

$$0 = \mu P_{n+1}^{(N)} - (\frac{\lambda}{n+1} + \mu) P_n^{(N)} + \frac{\lambda}{n} P_{n-1}^{(N)} \text{ for n=1,2,…,N-1 and} \tag{4}$$

$$0 = \frac{\lambda}{N} P_{N-1}^{(N)} - \mu P_N^{(N)} \tag{5}$$

Then the steady state probabilities of the system size are given by

$$P_n^{(N)} = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0^{(N)} \text{ for, } 1 \le n \le N \tag{6}$$

With $$P_0^{(N)} = \frac{1}{\left[1 + \sum_{n=1}^{N} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]} \tag{7}$$

**Proof**

We obtain the steady state probabilities by using iterative method.

solving(3),we have the value of $P_1$ as

$$P_1^{(N)} = \left(\frac{\lambda}{\mu}\right) P_0^{(N)} \tag{8}$$

Put n=1 in (4) and using (8) we get,

$$P_2^{(N)} = \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2 P_0^{(N)} \tag{9}$$

Similarly, put n=2 in (4) and using (9) we get,

$$P_3^{(N)} = \frac{1}{3!}\left(\frac{\lambda}{\mu}\right)^3 P_0^{(N)} \tag{10}$$

In general for n= 1,2,…,N, we can easily obtain

$$P_n^{(N)} = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0^{(N)}$$

Thus, the equation in (6) completely determine all the steady state probabilities

$P_n^{(N)}$ n=1,2,…,N.

To find $P_0^{(N)}$ ,using the normalization condition $\sum_{n=0}^{N} P_n^{(N)} = 1$ and the values $P_n^{(N)}$ , $1 \le n \le N$

$$\left(1+\sum_{n=1}^{N}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right)P_0^{(N)} = 1$$

Therefore,

$$P_0^{(N)} = \frac{1}{\left(1+\sum_{n=1}^{N}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right)}$$

Which is (7).

The steady state probabilities exist if

$$\left(1+\sum_{n=1}^{N}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right) < \infty \cdot$$

This completes the proof.

## 3.2 MEASURES OF EFFECTIVENESS

In this section, we derive some important measures. These can be used to study the performance of the queueing system under consideration.

**The Expected Number of Customers in the System**

$$L_s = \sum_{n=0}^{N} nP_n^{(N)}$$

$$= \sum_{n=1}^{N} n\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]P_0^{(N)}$$

**The Expected Number of Customers in the Queue**

$$L_q = \sum_{n=1}^{N}\left[\left(\frac{\lambda}{\mu}\right)^n\left[\frac{1}{(n-1)!}-\frac{1}{n!}\right]\right]P_0^{(N)}$$

**The Expected Waiting Time of Customer in the System**

$$W_s = \left[\frac{1}{\lambda}\sum_{n=1}^{N} n\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]\right]P_0^{(N)}$$

**The Expected Waiting Time of Customer in the Queue**

$$W_q = \frac{1}{\lambda}\left[\sum_{n=1}^{N}\left[\left(\frac{\lambda}{\mu}\right)^n\left[\frac{1}{(n-1)!}-\frac{1}{n!}\right]\right]P_0^{(N)}\right]$$

**The Expected Number of Customers Served**

$$\text{E[Customers Served]}=\sum_{n=1}^{N} n\mu P_n^{(N)}$$

$$=\sum_{n=1}^{N} n\mu\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]P_0^{(N)}$$

**Probability Distribution of Busy Period**

Prob(Busy period)= prob( $n\geq 1$)

$$=\sum_{n=1}^{N}\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]P_0^{(N)}$$

**Expected Number of Waiting Customers, Who Actually Wait,**

$$\text{E(Customer waiting)} = \frac{\sum_{n=2}^{N}(n-1)P_n^{(N)}}{\sum_{n=2}^{N}P_n^{(N)}}$$

$$=\frac{\sum_{n=2}^{N}(n-1)\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]P_0^{(N)}}{\sum_{n=2}^{N}\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]P_0^{(N)}}$$

## 3.3 INFINITE CAPACITY QUEUEWITH DISCOURAGEMENT

If the capacity of the system is infinite, then the system of differential difference equations of the model are in (1)

**Theorem 3.2**

If the steady state equations of the queueing model are

$$0=\mu P_1-\lambda P_0 \tag{11}$$

$$0=\mu P_{n+1}-\left(\frac{\lambda}{n+1}+\mu\right)P_n+\frac{\lambda}{n}P_{n-1},\ \ n\geq 1 \tag{12}$$

Then the steady state probabilities of the system are given by

$$P_n=\frac{e^{\left(-\lambda/\mu\right)}\left(\lambda/\mu\right)^n}{n!},\ \text{n=0,1,2,} \tag{13}$$

Where,

$$P_0=e^{\left(-\lambda/\mu\right)} \tag{14}$$

**Proof**

Similar proof in Theorem 3.1

## 3.4 MEASURES OF EFFECTIVENESS

In this section, we derive some important measures of infinite capacity discouraged queue are given

**The Expected Number of Customers in the System**

$$L_s = \sum_{n=0}^{\infty} nP_n$$

$$= \sum_{n=1}^{\infty} nP_n$$

$$= \left(\lambda/\mu\right) e^{\left(\lambda/\mu\right)} P_0$$

**The Expected Number of Customers in the Queue**

$$L_q = \sum_{n=1}^{\infty} (n-1)P_n$$

$$L_q = \left[\left\{\left(\lambda/\mu\right)-1\right\} e^{\left(\lambda/\mu\right)} + 1\right] P_0$$

**The Expected Waiting Time of Customer in the System**

$$W_s = \frac{e^{\left(\lambda/\mu\right)}}{\mu} P_0$$

**The Expected Waiting Time of Customer in the Queue**

$$W_q = \left(1/\lambda\right)\left[\left\{\left(\lambda/\mu\right)-1\right\} e^{\left(\lambda/\mu\right)} + 1\right] P_0$$

**The Expected Number of Customers Served**

$$\text{E[Customers Served]} = \sum_{n=1}^{\infty} n\mu P_n$$

$$= \lambda e^{\left(\lambda/\mu\right)} P_0$$

**Probability Distribution of Busy Period**

Prob(Busy period)= prob( $n \geq 1$)

$$= \left[e^{\left(\lambda/\mu\right)} - 1\right] P_0$$

## 4. INFINITE CAPACITY QUEUE WITHOUT DISCOURAGEMENT

Let $R_n(t)$, n=0, 1,2,… be the probability that there are n customers in the system at time t. Then, the forward Kolmogorov equations for this system are

$$\frac{d}{dt}\left(R_0(t)\right) = \mu R_1(t) - \lambda R_0(t),$$

$$\frac{d}{dt}\left(R_n(t)\right) = (n+1)\mu R_{n+1}(t) - (\lambda + n\mu)R_n(t) + \lambda R_{n-1}(t), \; n \geq 1 \tag{15}$$

Assume that initially the system is empty.

If the steady state equations of the queueing model are

$$0 = \mu R_1 - \lambda R_0 \tag{16}$$

$$0 = (n+1)\mu R_{n+1} - (\lambda + n\mu)R_n + \lambda R_{n-1}, \; n \geq 1 \tag{17}$$

and applying Theorem 3.2, then, we get, the steady state probabilities of the system are

$$P_n = \frac{e^{\left(-\lambda/\mu\right)}\left(\lambda/\mu\right)^n}{n!}, \text{ n=0,1,2,…}$$

and, $P_0 = e^{\left(-\lambda/\mu\right)}$ which is same as in (13) and (14).

Therefore, the steady state solution of discouraged arrivals queue with infinite capacity queue and the infinite capacity queuewithout discouragementare the same

## 5. NUMERICAL ILLUSTRATION

In table 1 values of the constant coefficients for N= 1,2,…,10 , $\lambda = 1$ and $\mu = 0.9$ are given. In table 2 values of the constant coefficients for N= 1,2,…,10 , $\lambda = 0.9$ and $\mu = 1$ are given for finite capacity queue with arrivals discouraged

**Table 1**

| N=1 | N=2 | N=3 | N=4 | N=5 | N=6 | N=7 | N=8 | N=9 | N=10 |
|---|---|---|---|---|---|---|---|---|---|
| 0.4737 | 0.3665 | 0.3382 | 0.3311 | 0.3295 | 0.3292 | 0.3292 | 0.3292 | 0.3292 | 0.3292 |
| 0.5263 | 0.4072 | 0.3758 | 0.3679 | 0.3661 | 0.3658 | 0.3658 | 0.3658 | 0.3658 | 0.3658 |
| | 0.2263 | 0.2087 | 0.2044 | 0.2034 | 0.2032 | 0.2032 | 0.2032 | 0.2032 | 0.2032 |
| | | 0.0773 | 0.0756 | 0.0754 | 0.0753 | 0.0753 | 0.0753 | 0.0753 | 0.0753 |
| | | | 0.0210 | 0.0209 | 0.0209 | 0.0209 | 0.0209 | 0.0209 | 0.0209 |
| | | | | 0.0047 | 0.0046 | 0.0046 | 0.0046 | 0.0046 | 0.0046 |
| | | | | | 0.0008 | 0.0008 | 0.0008 | 0.0008 | 0.0008 |
| | | | | | | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| | | | | | | | 0.0001 | 0.0001 | 0.0001 |

**Table 2**

| N=1 | N=2 | N=3 | N=4 | N=5 | N=6 | N=7 | N=8 | N=9 | N=10 |
|---|---|---|---|---|---|---|---|---|---|
| 0.5263 | 0.4338 | 0.4121 | 0.4075 | 0.4067 | 0.4066 | 0.4066 | 0.4066 | 0.4066 | 0.4066 |
| 0.4737 | 0.3905 | 0.3709 | 0.3668 | 0.3660 | 0.3659 | 0.3659 | 0.3659 | 0.3659 | 0.3659 |
| | 0.1757 | 0.1669 | 0.1650 | 0.1647 | 0.1647 | 0.1647 | 0.1647 | 0.1647 | 0.1647 |
| | | 0.0501 | 0.0495 | 0.0494 | 0.0494 | 0.0494 | 0.0494 | 0.0494 | 0.0494 |
| | | | 0.0111 | 0.0111 | 0.0111 | 0.0111 | 0.0111 | 0.0111 | 0.0111 |
| | | | | 0.0020 | 0.0020 | 0.0020 | 0.0020 | 0.0020 | 0.0020 |
| | | | | | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 |

For $\lambda = 1$ and $\mu = 0.9$ the values of the constant coefficients for infinite capacity queue in both arrivals discouraged and no discouraged are given in table 3. For $\lambda = 0.9$ and $\mu = 1$ the values of the constant coefficients are given in table 4

**Table 3**

| | |
|---|---|
| $P_0$ | 0.3292 |
| $P_1$ | 0.3658 |
| $P_2$ | 0.2032 |
| $P_3$ | 0.0753 |
| $P_4$ | 0.0209 |
| $P_5$ | 0.0046 |
| $P_6$ | 0.0008 |
| $P_7$ | 0.0001 |
| $P_8$ | 0.0001 |

**Table 4**

| | |
|---|---|
| $P_0$ | 0.4066 |
| $P_1$ | 0.3659 |
| $P_2$ | 0.1647 |
| $P_3$ | 0.0494 |
| $P_4$ | 0.0111 |
| $P_5$ | 0.0020 |
| $P_6$ | 0.0003 |

## 6. CONCLUSIONS

In this paper we studies a single sever queueing model with discouraged arrivals. We obtain the transient solution for discouraged arrival queues in both finite and infinite capacity. Finally a discouraged queueing system with infinite capacity is compared with infinite capacity queue without discouragementsystem which has the same steady state solution as the model considered.

## 7. REFERENCES

1. **S. I. Ammar , A.A. El-Sherbiny and R.O. Al-Seedy,** A Matrix Approach For The Transient Solution of an M/M/1/N Queue with discouraged arrivals and reneging. International Journal Of Computer Mathematics, 89,PP482-491, (2012).
2. **P. M. Morse,** Queues, Inventories And Maintenance, Willey, New York (1968).
3. **P. R. Parthasarathy,** A transient Solution to an M/M/1 Queue A simple Approach, Adv.Appl.Prob 19, PP 997-998 (1987).
4. **P.R. Parthasarathy and N.Selvaraju,** Transient Analysis Of A Queue Where Potential Customers Are Discouraged By Queue Length, Mathematical Problems In Engineering 7Pp 433-454, (2001).
5. **H.M.Srivastava and B.R.K. Kashyap,** Special FuntionsInQueueing Theory and Related Stochastic Process, Academic Press, New York (1982).
6. **K. Wang , Li. N and Z.Jiang,** Queueing System With Impatient Customers: A Review 2010 IEEE International Conference On Service Operations and Logistics And Infomatics 15-17 (2010), Shandong, pp. 82-87.